21/11/22

7. Realization spaces & universality

- understanding the multitude of polytopes has two aspects
  1) understanding the variety of combinatorial types
  2) understanding the variety in the realizations of particular combinatorial type

what we look at today

Questions:

- can I transform any two realizations into each other continuously?
- are there always particularly nice realizations of some sort?
  - with integer coordinates for vertices
  - inscribed in a sphere
  - as symmetric as the combinatorics permits

⟶ the lesson today is: almost everything can fail

⇝ "universality" is at the core of this

- things get wild as soon as
  - $d \geq 4$, (3-polytopes are "tame")
  - $n \geq d+4$ (last time we looked at $n-d \leq 3$)

7.1. Realization spaces

determines the dimension

- fix a combinatorial type (face lattice) $\mathcal{F}$

Def: the realization space $\mathcal{R}(\mathcal{F})$ is the "space" of all polytopes $P \subset \mathbb{R}^d$ with combinatorial type $\mathcal{F}$.

$$\mathcal{R}(\mathcal{F}) := \{ \underbrace{p : \mathcal{F}_0 \to \mathbb{R}^d}_{\in \mathbb{R}^{n \times d}} \mid \text{conv}\{p_v \mid v \in \mathcal{F}_0\} \text{ has combinatorial type } \mathcal{F} \}$$

n ... vertex count

- as a subset of $\mathbb{R}^{n\times d}$, $\mathcal{R}(\mathcal{F})$ comes with an induced topology
  - $\longrightarrow$ $\mathcal{R}(\mathcal{F})$ is a topological space associated to every combinatorial type, and we can ask typical topology questions.
- the definition of $\mathcal{R}(\mathcal{F})$ tells us basically nothing about the properties of this topological space

Questions:

- how does $\mathcal{R}(\mathcal{F})$ "look" like? How complicated can it be?
- is $\mathcal{R}(\mathcal{F})$ a manifold or not? ... what else could it be?
- what is the dimension of $\mathcal{R}(\mathcal{F})$? ... if this notion makes sense
- is $\mathcal{R}(\mathcal{F})$ connected?
  - ... simply connected?
  - ... contractable?
- can we compute its topological invariants?

- Euler characteristic
- Betti numbers
- fundamental group
- homology groups ...

- does $\mathcal{R}(\mathcal{F})$ contain rational points

NOTE: some of these questions make more sense when asked for the reduced realization space $\mathcal{R}(\mathcal{F}) / (\text{trivial transformations})$

transformations that preserve the combinatorial type but exist for every polytope

→ they unnecessarily "bloat" the realization space

e.g. - translations
- rotations
- general linear transformations
- projective transformations

- there exists a particularly nice model of the realization space that helps us understand better how it looks like

Centerered realization space $\mathcal{R}_0(\mathcal{F})$

= "space" of centered polytopes

:= P contains 0 in interior

Ex: $\mathcal{R}_0(\mathcal{F})$ is an open subset of $\mathcal{R}(\mathcal{F})$

→ captures all the essential properties of $\mathcal{R}(\mathcal{F})$

- Recall: there are two ways to describe a polytope
  1) $p_1, \dots, p_n \in \mathbb{R}^d \longrightarrow$ via its vertices
  2) $a_1, \dots, a_m \in \mathbb{R}^d \longrightarrow$ via its facet normals
- These descriptions are equivalent iff

$$\operatorname{conv}\{p_1, \dots, p_n\} = \underbrace{\{x \in \mathbb{R}^d \mid \langle x, a_i \rangle \leq 1 \ \forall i\}}_{\text{necessarily a polytope with } 0 \text{ in the interior}}$$

- We can force them to be equivalent via the vertex-facet incidences (which allow reconstruction of all of $\mathcal{F}$)
- This gives us an explicit definition of $\mathcal{R}_0(\mathcal{F})$:

$$\mathcal{R}_0(\mathcal{F}) := \left\{ (p, a) \in \mathbb{R}^{d \times (n+m)} \;\middle|\; \begin{array}{ll} \langle p_i, a_j \rangle = 1 & \text{if vertex } i \text{ lies in facet } j \\ \langle p_i, a_j \rangle < 1 & \text{otherwise} \end{array} \right\}$$

- $\mathcal{R}_0(\mathcal{F})$ is a semi-algebraic set

  := defined from polynomial equalities and strict inequalities

  (in fact, only quadratic polynomials)
- $\mathcal{R}_0(\mathcal{F})$ is a subset of $\mathbb{R}^{d \times (n+m)}$

  $\longrightarrow$ comes with an induced topology
- could be a manifold or not ...

  <u>BUT</u>: dimension is already well-defined for semi-algebraic sets
- Let's try to estimate $\underbrace{\dim \mathcal{R}_0(\mathcal{F})}_{= \dim \mathcal{R}(\mathcal{F})}$: DOFs $-$ constraints

$$\dim \mathcal{R}(\mathcal{F}) \geq \underbrace{d\cdot n + d\cdot m}_{\text{DOFs}} - \#\text{vertex-facet incidences}$$

only equality constraints lower the dimension

$$= d f_0 + d f_{d-1} - f_{0,d-1}$$

because some constraints could be redundant

element of the so-called flag f-vector

- $df_0 + df_{d-1} - f_{0,d-1}$ is known as the natural guess. (NG)
  - $\longrightarrow$ Is it accurate?

Examples:

- $\underline{d=2}$: $f_{0,1} = 2f_0 \longrightarrow NG = 2f_0 + 2f_1 - 2f_0 = \underline{2f_1}$
- $\underline{d=3}$: Ex: show $f_{0,2} = 2f_1$ (double counting)

$$\longrightarrow NG = 3f_0 + 3f_2 - 2f_1$$
$$= 3(f_0 - f_1 + f_2) + f_1$$
$$= f_1 + 6 \quad = \#\text{edges} + \text{translations} + \text{rotations} \;??$$

- let's first see when $\mathcal{R}(\mathcal{F})$ behaves nicely
- $\underline{d \leq 3}$:
  - $\mathcal{R}(\mathcal{F})$ is a manifold!
  - $\mathcal{R}(\mathcal{F})/_{(\ldots)}$ is contractible $\longrightarrow$ connected, simply connected ...

    the "isotropy conjecture" is satisfied

    (every realization can be continuously deformed into every other realization)
  - $\dim \mathcal{R}(\mathcal{F}) = NG \overset{d=3}{=} f_1 + 6$ (natural guess is correct)

- $\mathcal{F}$ can be realized with rational (therefore integer) vertex coordinates and facet normals
- $\mathcal{F}$ can be realized with all its combinatorial symmetries!

(- cannot always be realized inscribed in, say, a sphere)

- <u>simple / simplicial polytopes</u>: Ex: $\mathcal{R}(\mathcal{F}) \simeq \mathcal{R}(\mathcal{F}^\Delta)$
  - let's only consider simplicial
  - $f_{0,d-1} = d f_{d-1}$ (each facet is a simplex and contains d vertices)

$$\dim \mathcal{R}(\mathcal{F}) \overset{=}{\geq} d f_0 + d f_{d-1} - d f_{d-1} = \underline{d f_0}$$

* But this is already as large as it can get (moving every vertex independently and freely)

  - Likewise for simple polytopes $\mathcal{R}(\mathcal{F}) = d f_{d-1}$
  - $\mathcal{R}(\mathcal{F})$ is contractable, etc. ...

- In general many things can go wrong
  - $\mathcal{R}(\mathcal{F})$ might <u>not</u> be a manifold

    <u>Example</u>: $\mathcal{R}(24\text{-cell})$ has a singular point at the regular realization.
    But there it has dim. $48 = NG$

    <u>OPEN</u>: is the dimension everywhere 48?

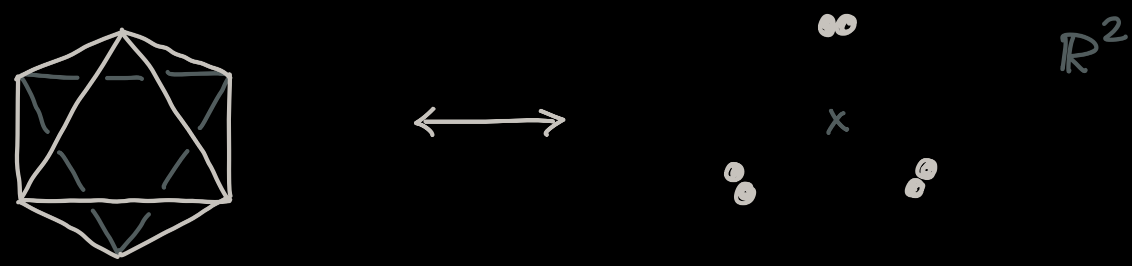
  - even if $\mathcal{R}(\mathcal{F})$ is a manifold, NG might be off.

    <u>Example</u>: bipyramid over Δ-prism
    $\dim \mathcal{R}(\mathcal{F}) = 26 > 25 = NG$

- Next we want to quantify the badness of $\mathcal{R}(\mathcal{F})$

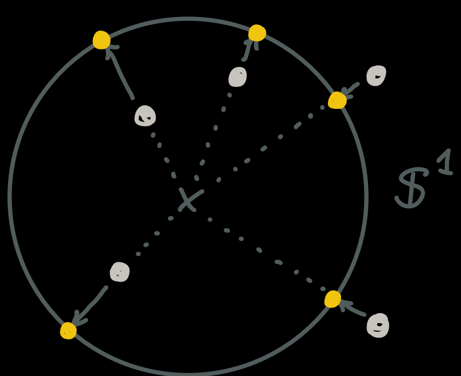
7.2. Gale diagrams & universality

- Recall: affine Gale duals



n-vertex d-polytope $\longleftrightarrow$ n-point $(n-d-1)$-arrangement

- Recall: spherical Gale diagrams



Obtained by projecting the non-zero points of the affine Gale dual onto the unit sphere

$\longrightarrow$ points are on a 1-dimensional curve, but the curve nees 2D

$\longrightarrow$ we can do better to loose one more dimension

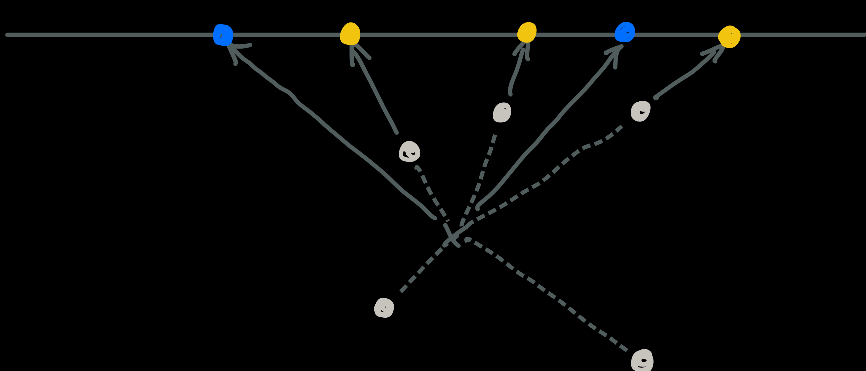
Def: (linear) Gale diagram

$\longrightarrow$ we instead project on a (non-central) hyperplane H

$+$

We also color the projections

- (later •) if they went straight onto H
- if they needed to pass by the origin

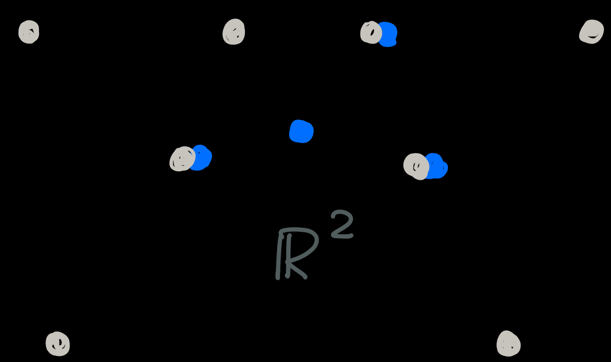


- If P is d-dimensional on n vertices, then the Gale diagram has dimension $n-d-2$
- in the following we only discuss 2D Gale diagrams

$$n-d-2=2 \longrightarrow n=d+4$$

- We will see: polytopes start to behave wildly at $d+4$ vertices because then their Gale diagrams are of sufficiently high dimension to encode complex behavior.

<u>Example</u>: a 2-dimensional Gale diagram of some 8-polytope



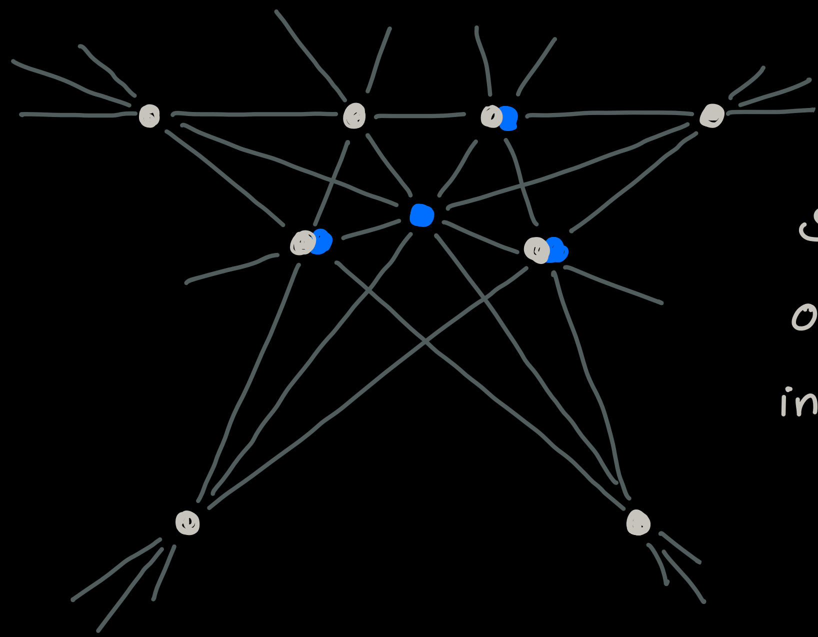
Lem:

(i) $q_1, \dots, q_n \in \mathbb{R}^2$ is a Gale diagram of a polytope iff no line separates all white (blue) points and a single blue (white) point from the rest.

(ii) each set of identical or colinear points in the Gale diagram encodes a facet of the polytope

$\longrightarrow$ changing the realization of a polytope continuously changes the Gale dual so that identical / colinear points stay identical / colinear

<u>Proof</u>: Ex



Some of the combinatorics of a polytope gets encoded in a point-line arrangement

Thm: (Mnëv)

2D point-line arrangements can encode arbitrary polynomial equations.

→ 2D point-line arrangements can compute arbitrary polynomials

→ d-polytopes with d+4 vertices can encode arbitrary polynomials

Idea: point-line arrangements can encode $a+b$ and $a \cdot b$, and therefore polynomials

→ see GeoGebra files

Thm (universality of polytopes)

For any semi-algebraic set S there is a polytope whose realization space is "essentially" S.

via so-called "stable equivalence"

NOTE: these polytopes have fairly high dimension

... BUT ...

Thm: (Richter-Gebert)

the same already holds for 4-polytopes!

(proof is more involved)

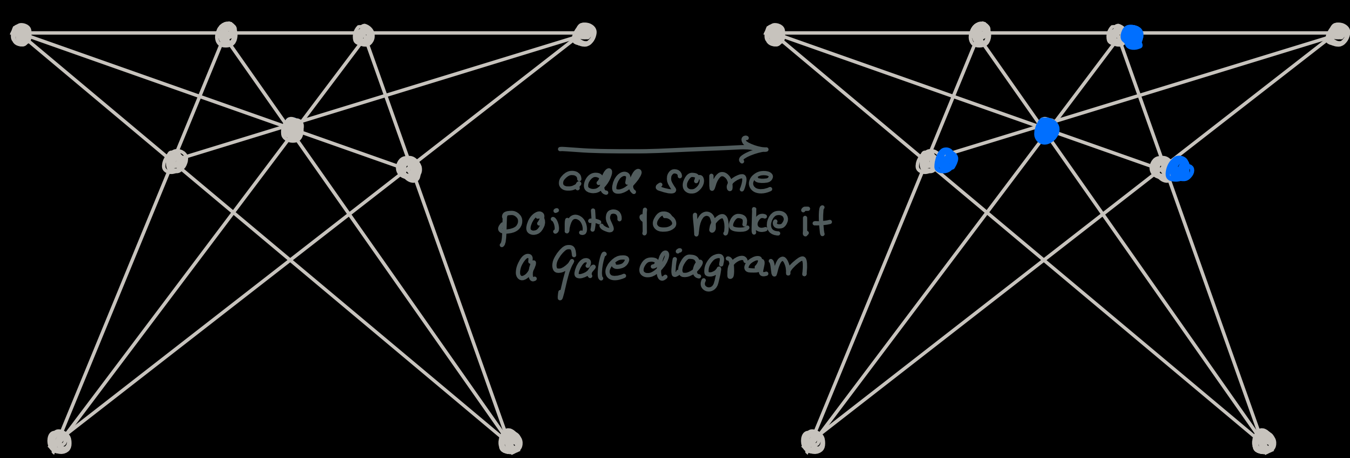
- These 4-polytopes have MANY vertices
  - ⟶ vertex - dimension tradeoff

- polytope realization spaces are as complicated as algebraic varieties := solution sets of polynomials in arbitrarily many variables

- How bad can they be?

7.3. Some weird polytopes

Perles configuration: has no completely rational realization



This gives a 12-vertex 8-polytope with no rational realization!

- *in fact*: no finite extension of $\mathbb{Q}$ is sufficient to represent every polytope.

OPEN: has the stellated 120-cell a rational realization?

extrude a pyramid from every dodecahedron facet until neighboring pentagonal pyramids become bipyramids

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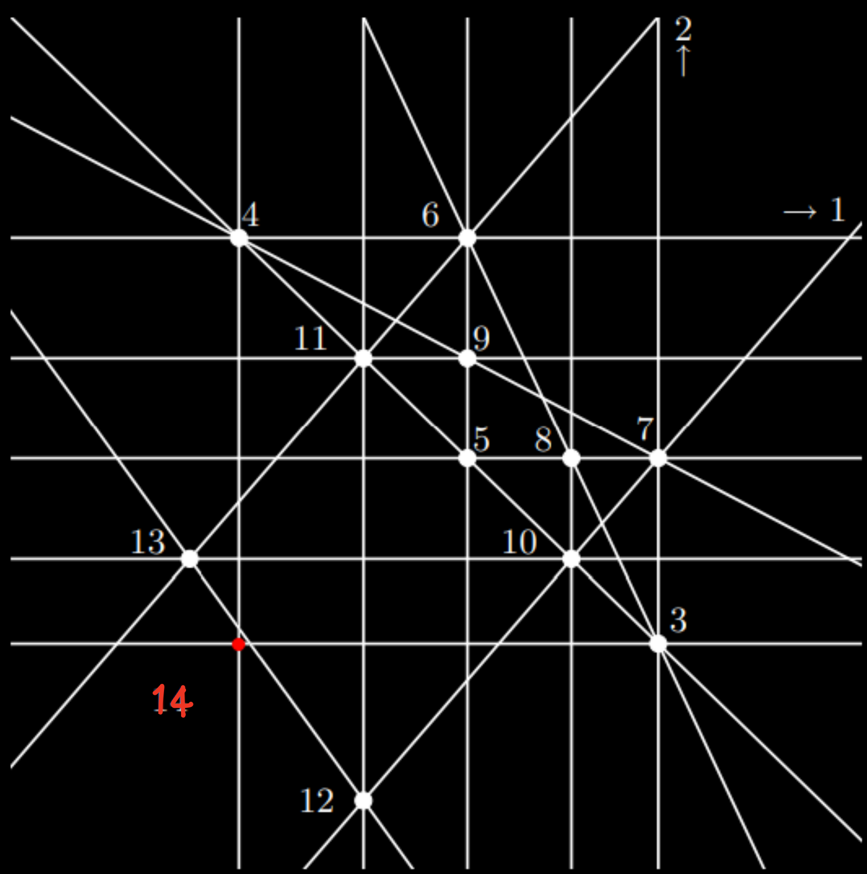
Richter-Gebert: 10-dimensional polytope on 14 vertices with a non-connected realization space



[Ziegler]

← The line $\overline{12,13}$ is not allowed to cross over the vertex 14, but there are realizations where it is on the other side.

OPEN: is there an example with fewer vertices?

→ this polytope violates the "isotopy conjecture"

→ there are two realizations that cannot be continuously morphed into each other.